AIAA JOURNAL
Vol. 43, No. 9, September 2005

Technical Notes

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Effect of Imperfections on Thermal Buckling of Functionally Graded Cylindrical Shells

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Nomenclature

m, n	=	number of half longitudinal and circumferential waves
N_{ij}	=	force resultant
T	=	temperature distribution
u, v, w	=	axial, circumferential, radial displacement under load
w^*	=	equivalent initial radial deviation from cylindrical shape
x, θ, z	=	axial, circumferential, radial coordinate direction
γ_{ij}	=	shear strain
ϵ_{ij}	=	normal strain
Θ	=	temperature difference with respect to the reference temperature
κ_{ij}	=	curvature of middle surface
μ	=	imperfection parameter
ϕ	=	Airy stress function

Introduction

BUCKLING experiments carried out on shell structures show that these structures are sensitive to the initial imperfections. Rigorous confirmation of the influence of initial imperfections was given by Koiter[1] in 1945. The Koiter analysis focuses attention on the initial postbuckling behavior and provides a theory that is exact at the bifurcation point.

The focus of research on thermal buckling of shell has changed gradually from metallic shells to composite ones. By the appearance of functionally graded materials (FGMs) in recent years, research on the buckling of shell structures entered a new area. Shahsiah and Eslami[2,3] presented a thermal buckling analysis of functionally graded cylindrical shells under several types of loadings. Recently, Shen[4] studied the postbucklig behavior of imperfect FGM cylindrical shells subjected to uniform temperature rise, using a singular perturbation technique.

Received 21 August 2004; revision received 7 March 2005; accepted for publication 8 March 2005.

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In the present Note, thermal buckling of simply supported imperfect FGM cylindrical shell is considered. The stability and compatibility equations for the imperfect FGM cylindrical shell are obtained, and the buckling analysis of shell under various thermal loads is carried out, using the traditional Galerkin's method, leading to closed-form solutions.

Fundamental Equations

Consider a thin circular cylindrical shell of mean radius R and thickness h with length L made of functionally graded material. The material properties follow the rule[5]

$$P(z) = P_{cm}[(2z+h)/2h]^{\xi} + P_m, \qquad P_{cm} = P_c - P_m \tag{1}$$

where P_m and P_c are the corresponding properties of the metal and ceramic, respectively, and ξ is the volume fraction exponent, which takes values greater than or equal to zero. In this analysis, the material properties such as Young's modulus $E(z)$, coefficient of thermal expansion $\alpha(z)$, and thermal conductivity $k(z)$ may be expressed by Eq. (1), where Poisson's ratio ν is considered to be constant across the thickness.

The Koiter model for the axisymmetric geometrical imperfection of cylindrical shell is expressed as (see Ref. 6)

$$w^* = -\mu h \cos(m\pi x/L), \qquad -L/2 \le x \le +L/2 \tag{2}$$

where μh is the amplitude of imperfection of the shell middle surface $(0 \le \mu \le 1)$.

Based on the first-order classical shell theory and using the Sanders nonlinear kinematic relations, the equilibrium equations of an imperfect functionally graded cylindrical shell may be derived as

$$N_{xx,x} + N_{xy,y} = 0, \qquad N_{xy,x} + N_{yy,y} = 0$$

$$(D - B^2/C)\nabla^4 w + N_{yy}/R - \left[N_{xx}(w+w^*)_{,xx} + N_{yy}w_{,yy} + 2N_{xy}w_{,xy}\right] = 0 \tag{3}$$

where $y = R\theta$ and

$$N_{xx} = C(\epsilon_{xx} + \nu\epsilon_{yy}) + B(\kappa_{xx} + \nu\kappa_{yy}) - T_0$$

$$N_{yy} = C(\epsilon_{yy} + \nu\epsilon_{xx}) + B(\kappa_{yy} + \nu\kappa_{xx}) - T_0$$

$$N_{xy} = [C(1-\nu)/2]\gamma_{xy} + B(1-\nu)\kappa_{xy} \tag{4}$$

$$\epsilon_{xx} = u_{,x} + \tfrac{1}{2}w_{,x}^2 + w_{,x}^* w_{,x}, \qquad \kappa_{xx} = -(w+w^*)_{,xx}$$

$$\epsilon_{yy} = v_{,y} + w/R + \tfrac{1}{2}w_{,y}^2, \qquad \kappa_{yy} = -w_{,yy}$$

$$\gamma_{xy} = v_{,x} + u_{,y} + (w+w^*)_{,x}w_{,y}, \qquad \kappa_{xy} = -w_{,xy} \tag{5}$$

Here, C, B, D, and T_0 are defined as

$$C = \frac{h}{1-\nu^2}\left(E_m + \frac{E_{cm}}{\xi+1}\right), \qquad B = \frac{h^2 E_{cm}}{2(1-\nu^2)}\left(\frac{\xi}{\xi^2+3\xi+2}\right)$$

$$D = \frac{h^3}{4(1-\nu^2)}\left[\frac{E_m}{3} + \frac{E_{cm}(\xi^2+\xi+2)}{\xi^3+6\xi^2+11\xi+6}\right]$$

$$T_0 = \frac{1}{1-\nu}\int_{-h/2}^{+h/2} E\alpha\Theta\,\mathrm{d}z \tag{6}$$

The stability equation of a FGM cylindrical shell may be obtained through the second variation of the potential energy function, which is

$$N_{xx1,x} + N_{xy1,y} = 0, \qquad N_{xy1,x} + N_{yy1,y} = 0$$

$$(D - B^2/C)\nabla^4 w_1 + N_{yy1}/R - \left[N_{xx1}\left(w_0 + w^*\right)_{,xx} + 2N_{xy1}w_{0,xy} + N_{yy1}w_{0,yy} + N_{xx0}w_{1,xx} + N_{yy0}w_{1,yy} + 2N_{xy0}w_{1,xy}\right] = 0 \tag{7}$$

where w_0 is related to the equilibrium configuration, w_1 is arbitrary small neighboring increment, and N_{ij1} are the force resultants related to the neighboring state.

When the Airy stress functions $N_{xx1} = \phi_{,yy}$, $N_{yy1} = \phi_{,xx}$, and $N_{xy1} = -\phi_{,xy}$, are introduced, the first and second stability equations are automatically satisfied and the third stability equation reduces to

$$(D - B^2/C)\nabla^4 w_1 + \phi_{,xx}/R - \left[\phi_{,yy}\left(w_0 + w^*\right)_{,xx} - 2\phi_{,xy}w_{0,xy} + \phi_{,xx}w_{0,yy} + N_{xx0}w_{1,xx} + N_{yy0}w_{1,yy} + 2N_{xy0}w_{1,xy}\right] = 0 \tag{8}$$

The compatibility equation in terms of the Airy stress function and the lateral displacement is

$$\nabla^4\phi - C(1-\nu^2)\left[w_{1,xx}/R - \left(w_0 + w^*\right)_{,xx}w_{1,yy} + 2w_{0,xy}w_{1,xy} - w_{1,xx}w_{0,y}\right] = 0 \tag{9}$$

Buckling Analysis

Consider an imperfect functionally graded cylindrical shell with simply supported edge conditions subjected to thermal load. For the axisymmetric configurations on the primary path, $u_0 = u_0(x)$, $v_0 \equiv 0$, and $w_0 = w_0(x)$ (Ref. 6). When these expressions are used and it is assumed that the temperature distribution is independent of x and y, the prebuckling coefficients are obtained by the solution of the equilibrium equations as

$$N_{xx0} = -T_0, \qquad N_{yy0} = C(1-\nu^2)(w_0/R) - T_0, \qquad N_{xy0} = 0$$

$$w_0 = T_0R\big/C(1-\nu^2) + q\cos(m\pi x/L) \tag{10}$$

where

$$q = -\mu h\frac{T_0(m\pi/L)^2}{(D - B^2/C)(m\pi/L)^4 - T_0(m\pi/L)^2 + C(1-\nu^2)/R^2} \tag{11}$$

Introducing Eqs. (2) and (10) into Eqs. (8) and (9) and considering $w_0 = w_0(x)$ results in the coupled linear equations of stability and compatibility as

$$(D - B^2/C)\nabla^4 w_1 + \phi_{,xx}/R - \left\{-T_0w_{1,xx} + [C(1-\nu^2)/R]q\cos(m\pi x/L)w_{1,yy} + \phi_{,yy}(\mu h - q)(m\pi/L)^2\cos(m\pi x/L)\right\} = 0$$

$$\nabla^4\phi - C(1-\nu^2)\left[w_{1,xx}/R - w_{1,yy}(\mu h - q) \times (m\pi/L)^2\cos(m\pi x/L)\right] = 0 \tag{12}$$

To solve the system of Eqs. (12), with the consideration of the simply supported boundary conditions, the approximate solutions may be considered as[6]

$$w_1 = \alpha_{mn}\cos(m\pi x/L)\cos(ny/R)$$

$$\phi = \beta_{mn}\cos(m\pi x/L)\cos(ny/R) \tag{13}$$

where α_{mn} and β_{mn} are constant coefficients that depend on m and n, where $m, n = 1, 2, \ldots$. Substituting the approximate solutions (13) into Eqs. (12) gives the residues, R_1 and R_2. According to Galerkin's method, R_1 and R_2 are made orthogonal with respect to the approximate solutions (13), and then the determinant of the obtained system of equations for the coefficients α_{mn} and β_{mn} is set to zero, which for $m = 4k \pm 1$ yields

$$s_0{T_0}^3 + s_1{T_0}^2 + s_2T_0 + s_3 = 0 \tag{14}$$

where $s_0, \ldots, s_3$ are functions of m, n, L, R, h, and μ and the material properties of shell. For $m = 4k$, the determinant of coefficients has no result.

Thermal Buckling

Uniform Temperature Rise

Consider a cylindrical shell under uniform temperature T_i. For a simply supported edge conditions, where the axial displacement is prevented, the uniform temperature may be raised to T_f such that the shell buckles. Using the last of Eqs. (6), we have

$$\Theta = T_f - T_i = \Delta T$$

$$T_0 = \frac{\Delta T h}{1-\nu}\left[E_m\alpha_m + \frac{E_m\alpha_{cm} + E_{cm}\alpha_m}{\xi+1} + \frac{E_{cm}\alpha_{cm}}{2\xi+1}\right] \tag{15}$$

Nonlinear Temperature Through the Thickness

Consider a FGM cylindrical shell in which the temperature of the inner and outer surfaces are T_m and T_c , respectively. Solving the heat conduction equation across the shell thickness and then using the last of Eqs. (6) gives $T(z)$ and T_0, respectively, as

$$T(z) = T_m + \Delta T\left(\frac{2z+h}{2h}\right) \times \left(\sum_{s=0}^{\infty}\left\{\left[-\frac{k_{cm}}{k_m}\left(\frac{2z+h}{2h}\right)^{\xi}\right]^s\Big/(\xi s+1)\right\}\Big/\sum_{s=0}^{\infty}\left[\left(\frac{-k_{cm}}{k_m}\right)^s\Big/(\xi s+1)\right]\right), \qquad \Delta T = T_c - T_m$$

$$T_0 = \frac{\Delta T h}{1-\nu}\left\{\sum_{s=0}^{\infty}\left(-\frac{k_{cm}}{k_m}\right)^s\left(\frac{1}{\xi s+1}\right) \times \left(\frac{E_m\alpha_m}{\xi s+2} + \frac{E_m\alpha_{cm} + E_{cm}\alpha_m}{\xi(s+1)+2} + \frac{E_{cm}\alpha_{cm}}{\xi(s+2)+2}\right)\Big/\sum_{s=0}^{\infty}\left[\left(-\frac{k_{cm}}{k_m}\right)^s\Big/(\xi s+1)\right]\right\} \tag{16}$$

For shells with isotropic material, the solution of the heat conduction equation is linear. For a very thin cylindrical shell of FGM, the linear temperature assumption may be justified, too. In this latter case, we have

$$T(z) = \Delta T[(2z+h)/2h] + T_m, \qquad \Delta T = T_c - T_m$$

$$T_0 = [\Delta T h/(1-\nu)]\{E_m\alpha_m/2 + (E_m\alpha_{cm} + E_{cm}\alpha_m)/(\xi+2) + E_{cm}\alpha_{cm}/(2\xi+2)\} \tag{17}$$

The critical temperature difference, in which thermal buckling occurs, may be written from Eqs. (15–17) with a single equation:

$$\Delta T_{\mathrm{cr}} = (1-\nu)T_{0\,\mathrm{min}}/h\psi \tag{18}$$

where ΔT_{cr} is the critical temperature difference and $T_{0\,\mathrm{min}}$ is obtained by minimizing the solutions of Eq. (14) with respect to m and n.

Result and Discussion

Let us assume $\mu = 0$, which corresponds to the equations for a perfect cylindrical shell. For $\xi = 1$ and in the case of uniform temperature rise, the solution may be validated with the solution obtained by Shen,[4] for two types of FGM namely, Si_3N_4/SUS304 and $ZrO_2/Ti - 6Al - 4V$ with temperature-independent properties given in Ref. 4, as shown in Table 1. If, in addition, we take $P_c = P_m$ (pure metallic shell), the solution may be validated with the closed-form solution obtained by Eslami et al.[7] for an isotropic cylindrical shell as shown in Table 1. In both cases the comparison is well justified.

The combination of materials is assumed to consist of aluminum and alumina for the following cases. Young's modulus, Poisson's ratio, thermal conductivity, and the coefficient of thermal expansion are, for aluminum, 70 GPa, 0.3,204 W/m · K, and 23.0×10^{-6}/°C, and for alumina, 380 GPa, 0.3, 10.4 W/m · K, and 7.4×10^{-6}/°C, respectively.

Variation of the ratio of critical temperature difference for the imperfect cylinder to the critical temperature difference for the corresponding perfect cylinder vs the imperfection parameter μ for a FGM shell under uniform temperature rise is shown in Fig. 1. As the amplitude of imperfection increases, the thermal buckling ratio decreases. The curve is independent of the values of the volume fraction exponent ξ.

The influences of cylindrical shell geometry on buckling load ΔT_{cr} are shown in Fig. 2. Figure 2a shows the thermal buckling load vs h/R, when $L/R = 1$ and $\mu = 0.5$, for the uniform temperature rise. As the ratio h/R increases, the buckling load increases. Figure 2b is the variation of thermal buckling load for the uniform temperature rise vs L/R, when $h/R = 0.005$ and $\mu = 0.5$. Bucking load slowly increases as the ratio L/R increases.

Figure 3 shows a comparison between the buckling loads of three loading cases, uniform temperature rise (UTR), nonlinear temperature distribution (NTD), and linear temperature distribution (LTD) across the thickness, for metallic and FGM ($\xi = 1$) imperfect cylindrical shells, vs imperfection parameter μ. For the FGM shell, the thermal buckling load of UTR is the lowest and the NTD load is the highest curve. For the pure metallic shell, NTD and LTD curves coincide, as expected, and are larger than the UTR curve.

In Figs. 1–3, the minimum value of number of longitudinal and circumferential buckling half waves was often found to be $m_{min} = 1$ and $n_{min} = 7$, 8, 9, and 11.

Table 1 Comparisons of buckling temperature difference ΔT_{cr} for perfect FGM and perfect isotropic cylindrical shells with the known data in the literature[a]

Material	Ref. 4	Present study	Ref. 7 (pure metal)	Present study (pure metal, $P_c = P_m$)
Si_3N_4/SUS304	118.4561	118.7456	139.8638	139.7514
$ZrO_2/Ti - 6Al - 4V$	99.5912	99.4009	85.9692	85.9001

[a]Where $h/r = 0.025$, $L/R = 0.866$, and $\xi = 1$.

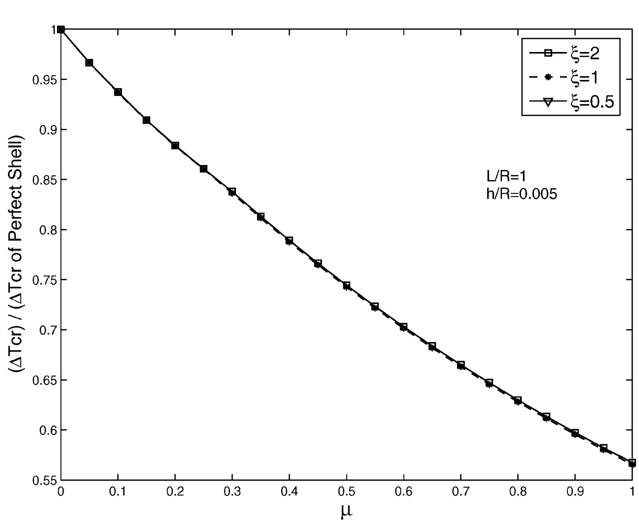

Fig. 1 Influence of imperfection magnitude.

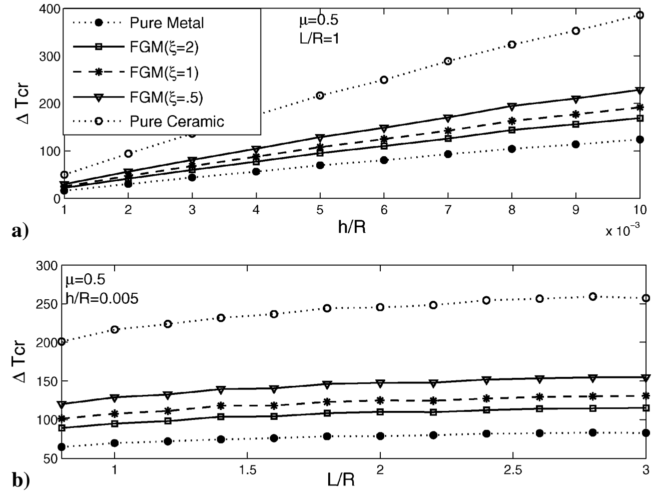

Fig. 2 Influence of shell geometry.

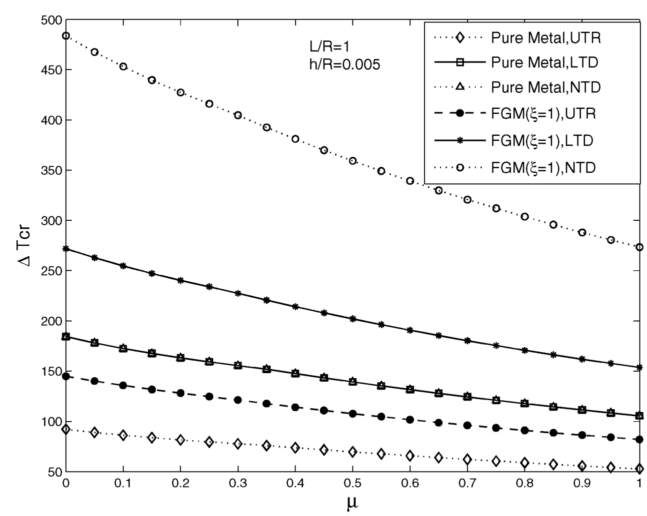

Fig. 3 Comparison between different types of thermal loads.

Conclusions

The thermal buckling analysis of imperfect functionally graded cylindrical shells under three types of thermal loadings are investigated. The following conclusions were reached:

1) A functionally graded cylindrical shell is sensitive to imperfection. The buckling load of an imperfect functionally graded cylindrical shell is considerably lower than the buckling load of a perfect one.

2) Decrease of the thermal buckling strength of an imperfect FGM cylindrical shell is independent of the volume fraction exponent.

3) The critical temperature difference ΔT_{cr} for the imperfect functionally graded cylindrical shell generally increases with increasing the shell thickness.

4) The critical temperature difference ΔT_{cr} for the imperfect functionally graded cylindrical shell is approximately independent of the length of the shell.

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